第三屆校際系統建模與優化競賽

The 3rd Interschool Competition on System Modeling & Optimization (COSMO)

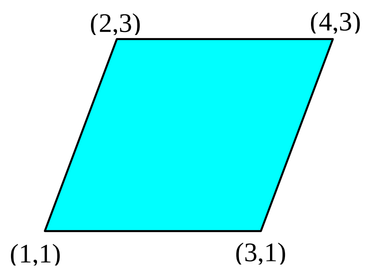
10 May 2008

Solution Sketch

The fourth term Legislative Council election will be held on September 7, 2008, and the Registration and Electoral Office (REO) is now actively preparing for it. One of the most important issues facing the REO is the setting up of polling stations. In order to reduce cost, the number of polling stations should not be too large. On the other hand, each polling station should cover as many voters in its corresponding district as possible. If you were a member of the REO, how would you set up the polling stations?

Question 1

To tackle this problem, let us begin with a simple example. Suppose that a district can be represented as a polygon; see the figure below, where the numbers represent the coordinates of the vertices of the polygon (unit: kilometers):



Suppose that whether a voter in the district would vote depends only on how far the polling station is from her. In order words, if the polling station is sufficiently close to her, then she would vote. Otherwise, she would not vote. For this reason, the REO takes certain measures to encourage voters to vote. For instance, the REO places advertisements on major media and arranges transportation for the voters. As a result of REO's efforts, voters who live within a distance r from the polling station will vote, where r is a parameter that is directly affected by REO's actions. Clearly, the REO would like to maximize r. However, in order to reduce cost, the REO decides that each polling station will only serve the voters in one district. In order words, the coverage of the polling station should lie within the boundary of the district.

For the figure shown above, where would you place the polling station so that its coverage is maximized and at the same time lies within the boundary of the district? What is the coverage radius r?

SOLUTION: For the figure shown above, intuition tells us that the center of the largest circle inscribed in the parallelogram should coincide with the center of the parallelogram, i.e. at *(2.5, 2)*, and the radius of the circle is $2/\sqrt{5}$. However, a more rigorous approach is the following. Let *(a,b)* be the center of the largest inscribed circle, and let *r* be its radius.

Note that the sides of the parallelogram are given by the following linear inequalities:

$$y \geq 1$$
$$y \leq 3$$
$$-2x + y \leq -1$$
$$-2x + y \geq -5$$

Now, recall that the distance between the point *(a,b)* to the line *cx+dy+e=0* is given by:

$$\frac{|ac + bd + e|}{\sqrt{c^2 + d^2}}$$

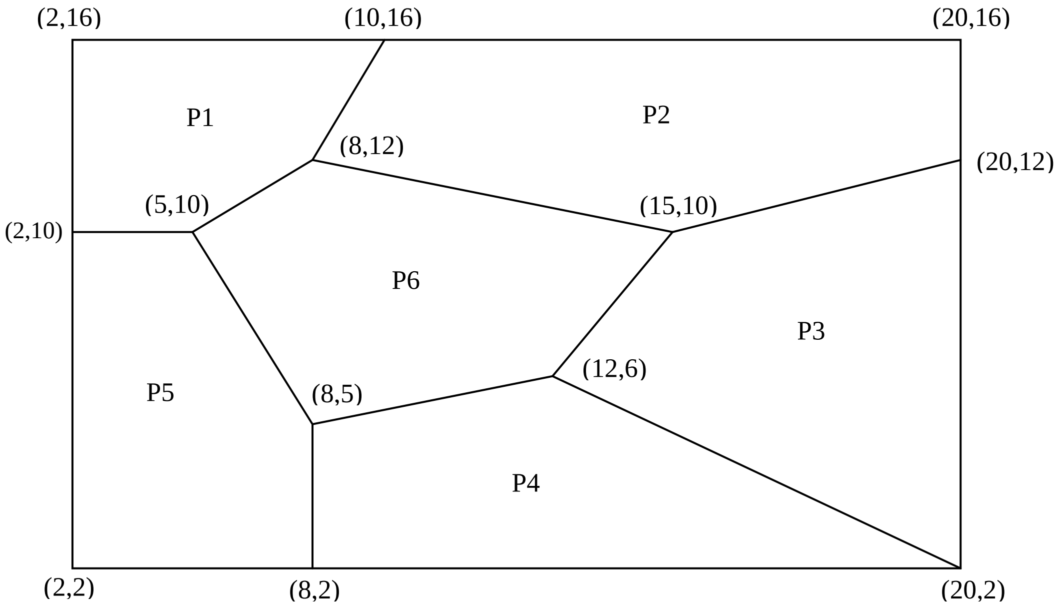
Since *(a,b)* lies inside the parallelogram, the variables *a,b,r* must satisfy the following constraints:

$$b - 1 \geq r$$
$$3 - b \geq r$$
$$\frac{2a - b - 1}{\sqrt{5}} \geq r$$
$$\frac{-2a + b + 1}{\sqrt{5}} \geq r$$

and our goal is to maximize *r*.

Question 2

Suppose that the figure below is a subdivision of the Shatin district: (unit: kilometers)



Each of the polygons *P1,…,P6* represents a district. Similar to Question 1, how would you place the polling stations in these six districts so that the coverage of each polling station is maximized and at the same time lies within its corresponding district?

If you cannot solve Question 2, you may guess the locations of the polling stations and use the results for the following question.

SOLUTION: It suffices to determine the linear inequalities defining each polygon and then use the technique outlined in Question 1 to solve the problem. We obtain:
P1=(4.7889, 13.2111)
P2=(14.8424, 13.0809)
P3=(16.824, 7.1388)
P4=(11.7741, 3.94188)
P5=(4.81976, 4.81976)
P6=(9.43338, 8.4605)

Question 3

A very important group of people in an election is of course the candidates. Suppose that a candidate from the Shatin district would like to visit all the six polling stations to thank her supporters. Based on your answer to Question 2, can you determine the order in which the candidate should visit the polling stations, so that the total distance traveled is minimized?

SOLUTION: Once the coordinates of the six polling stations are obtained, their pairwise distances are also known. Let $p_1,\ldots,p_6$ be the coordinates of the six polling stations. Then, it suffices to find an arrangement of the six numbers 1,2,…,6, say *i1,…,i6,* such that:

$$d\left(p_{i1}-p_{i2}\right)+d\left(p_{i2}-p_{i3}\right)+d\left(p_{i3}-p_{i4}\right)+d\left(p_{i4}-p_{i5}\right)+d\left(p_{i5}-p_{i6}\right)$$

is minimized. Here, $d(p_{i1}-p_{i2})$ is the distance between *i1* and *i2*.